Multishot Codes for Network Coding using Rank-Metric Codes

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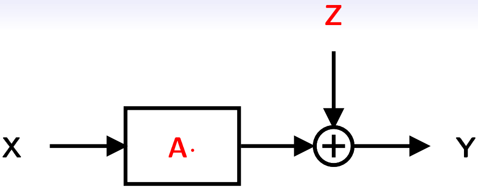
Introduction

The matrix channel



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- Input, output: $\mathbf{X}, \mathbf{Y} \in \mathbb{F}_q^{N \times T}$
- Transfer matrix: $\mathbf{A} \in \mathbb{F}_q^{N \times N}$
  - Unknown in a noncoherent scenario
  - Depends on network topology and coding coefficients
- Error matrix: $\mathbf{Z} \in \mathbb{F}_q^{N \times T}$
  - Related to link errors

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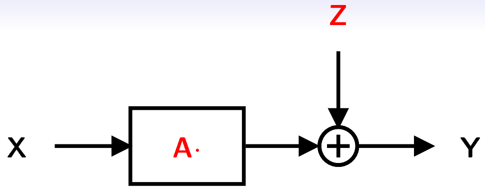

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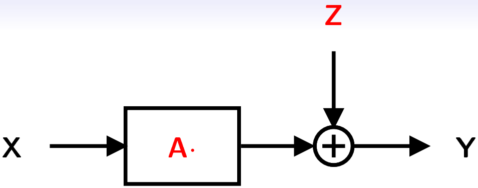

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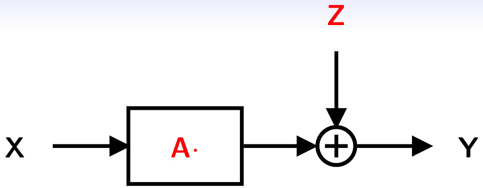

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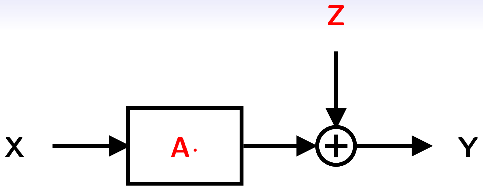

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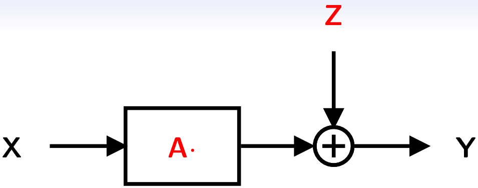

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- Subspace coding [Koetter-Kschischang '08]
  - Encode information in the subspace spanned by the rows of $\mathbf{X}$
- Lifting [Silva-K-K '08] a.k.a. channel training [Yang-H-M-Y-Y '10]
  - Special case of subspace coding
  - Roots on Chou's pratical network coding [Chou-W-K '03]
- Approaches (error model)
  - Combinatorial (algebraic)
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One-shot vs. multishot

- Mainly so far: one-shot codes
  - Use the channel only once
  - Okay if T can grow
- Proposed [Koetter-Kschischang '08] but not explored: multishot codes
  - Use the channel many times

This work: Block codes [Nobrega-Uchoa '09]

Transmit sequences of n matrices (or subspaces):

$$(\mathbf{X}_0, \ldots, \mathbf{X}_{n-1})$$

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Why use multishot codes?

- Two results:
  - **Channel capacity** can be achieved for large T [Silva-K-K '10]
  - **Combinatorial codes** are nearly optimal for large T [Silva-K '09]
- What if packet size T cannot be made too large?
  - For example: fast topology-changing networks
- Under this condition:
  - **Information theory**: use the channel many times to achieve capacity [Shannon!] [Jafari-M-F-D '09] [Yang-H-M-Y-Y '10]
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- Network fading:
  - Multishot can correct entirely lost shots (intershot redundancy)

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Combinatorial results

One-shot [Silva-K-K '08]

Let $\mathcal{X} \subseteq \mathbb{F}_q^{N \times T}$ be an one-shot code. If

$$\operatorname{rankdef} \mathbf{A} \leq \rho \quad \text{and} \quad \operatorname{rank} \mathbf{Z} \leq \tau$$

then codeword $\mathbf{X}$ can be decoded from $\mathbf{Y}$ provided

$$d_S(\langle \mathcal{X} \rangle) > 2(\rho + 2\tau)$$

where $\langle \mathcal{X} \rangle$ is the (one-shot) subspace code obtained from $\mathcal{X}$ and

$$d_S(U, V) \triangleq \dim(U \dotplus V) - \dim(U \cap V)$$

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Multishot

Let $\boldsymbol{\mathcal{X}} \subseteq (\mathbb{F}_q^{N \times T})^n$ be a multishot code. If

$$\sum_{j=0}^{n-1} \text{rankdef}\, \mathbf{A}_j \leq \rho \quad \text{and} \quad \sum_{j=0}^{n-1} \text{rank}\, \mathbf{Z}_j \leq \tau$$

then codeword $(\mathbf{X}_0, \ldots \mathbf{X}_{n-1})$ can be decoded from $(\mathbf{Y}_0, \ldots \mathbf{Y}_{n-1})$ provided

$$d_S(\langle \boldsymbol{\mathcal{X}} \rangle) > 2(\rho + 2\tau)$$

where $\langle \boldsymbol{\mathcal{X}} \rangle$ is the (multishot) subspace code obtained from $\boldsymbol{\mathcal{X}}$ and

$$d_S(\boldsymbol{U}, \boldsymbol{V}) \triangleq \sum_{j=0}^{n-1} d_S(U_j, V_j)$$

Rank-Metric Codes

Definitions and results

- Studied by Gabidulin [Gabidulin '85] and Roth [Roth '91]
- Rank-metric codes: $\mathcal{R} \subseteq \mathbb{F}_{q^M}^N$
  - An (N, K, D) code over an extended field $\mathbb{F}_{q^M}$
- Metric of concern: rank distance:

$$d_R(\mathbf{u}, \mathbf{v}) = \text{rank}(\underline{\mathbf{v}} - \underline{\mathbf{u}})$$

where $\mathbf{u} \in \mathbb{F}_{q^M}^N$ is translated to $\underline{\mathbf{u}} \in \mathbb{F}_q^{N \times M}$

- Singleton bound is valid:

$$D \leq N - K + 1$$

achieved with Gabidulin codes [Gabidulin '85] provided $N \leq M$

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Application to network coding

Rank-metric code $\longrightarrow$ Matrix code for network coding

Lifting operation [Silva-K-K '08]

Identity matrix in the header:

$$\begin{aligned} \mathcal{I}: \mathbb{F}_{q^M}^{N} &\longrightarrow \mathbb{F}_q^{N \times T} \\ \mathbf{u} &\longmapsto \left[\ \mathbf{I} \ \middle|\ \underline{\mathbf{u}}\ \right] \end{aligned}$$

with $T = N + M$

- Key result:

$$d_S(\langle \mathcal{I}(\mathbf{u}) \rangle, \langle \mathcal{I}(\mathbf{v}) \rangle) = 2\, d_R(\mathbf{u}, \mathbf{v})$$

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Extension to multishot

- Multishot rank-metric codes: $\mathcal{R} \subseteq (\mathbb{F}_{q^M}^N)^n$
  - Codewords are n-sequences of N-tuples over $\mathbb{F}_{q^M}$:

$$\boldsymbol{u} = (\mathbf{u}_0, \ldots, \mathbf{u}_{n-1}) \text{ where each } \mathbf{u}_j \in \mathbb{F}_{q^M}^N$$

- Metric: extended rank distance:

$$\mathrm{d_R}(\boldsymbol{u}, \boldsymbol{v}) = \sum_{j=0}^{n-1} \mathrm{d_R}(\mathbf{u}_j, \mathbf{v}_j),$$

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Extended lifting

Multishot rank-metric code $\longrightarrow$ Multishot matrix code

Extended lifting operation

Lifting of each element:

$$\begin{aligned} \mathcal{I} : (\mathbb{F}_{q^M}^{N})^n &\longrightarrow (\mathbb{F}_q^{N \times T})^n \\ (\mathbf{u}_0, \ldots, \mathbf{u}_{n-1}) &\longmapsto (\mathcal{I}(\mathbf{u}_0), \ldots, \mathcal{I}(\mathbf{u}_{n-1})) \end{aligned}$$

with $T = N + M$

- Result:

$$\mathrm{d_S}(\langle \mathcal{I}(\boldsymbol{u}) \rangle, \langle \mathcal{I}(\boldsymbol{v}) \rangle) = 2\, \mathrm{d_R}(\boldsymbol{u}, \boldsymbol{v})$$

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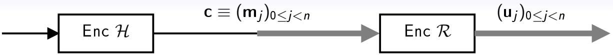
Multilevel Construction

Concatenated codes



Ingredients

1. A p-ary (n, k, d) outer code $\mathcal{H}$ (typically Reed-Solomon)
2. A Q-ary (N, K, D) inner code $\mathcal{R}$
    - Originally binary: $Q = 2$
    - Here rank-metric: $Q = q^M$

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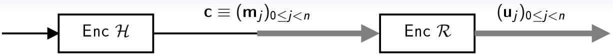

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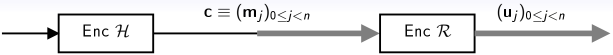

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- We have:

$$\mathbf{c} = (c_0, \ldots, c_{n-1}) \equiv (\mathbf{m}_0, \ldots, \mathbf{m}_{n-1})$$

where each $c_j \in \mathbb{F}_p = \mathbb{F}_{Q^K}$ is translated to $\mathbf{m}_j \in \mathbb{F}_Q^K$

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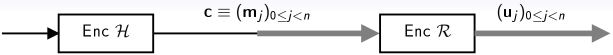

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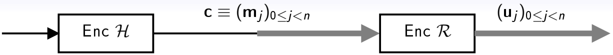
- There are:
  - 1 outer encoding: to form $\mathbf{c}$
  - n inner encodings: each $\mathbf{m}_j \in \mathbb{F}_Q^K$ is encoded to $\mathbf{u}_j \in \mathbb{F}_Q^N$

Concatenated codes



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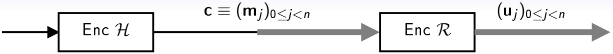
- Result: a Q-ary (nN, kK) concatenated code with

$$d_{\min} \geq dD$$

or…

Concatenated codes



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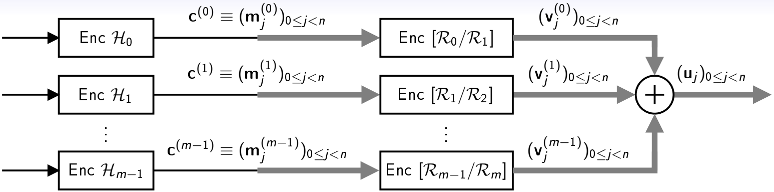
- Result: a multishot code $\boldsymbol{\mathcal{R}} \subseteq (\mathbb{F}_{q^M}^N)^n$ with $|\boldsymbol{\mathcal{R}}| = |\mathcal{H}|$ and

$$d_{\min} \geq dD$$

Generalized concatenated codes



- Introduced by Blokh and Zyablov [Blokh-Zyablov '74]
- Also known as multilevel concatenated codes

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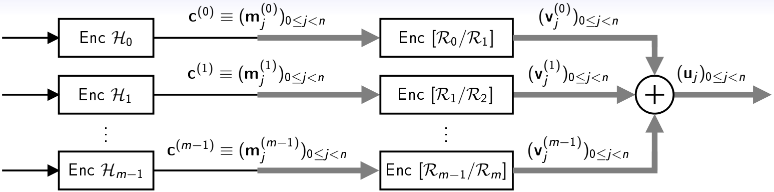

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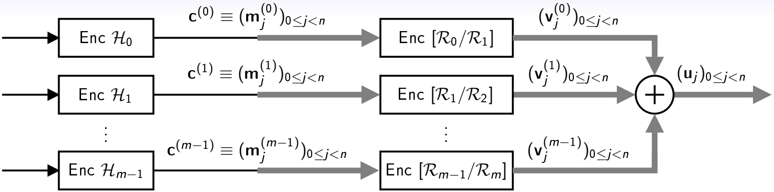

Ingredients

1. m outer codes $\mathcal{H}_0, \ldots, \mathcal{H}_{m-1}$, (n_i, k_i, d_i), p_i-ary
2. $m+1$ subcodes $\mathcal{R}_0 \supseteq \cdots \supseteq \mathcal{R}_m$ of a parent code $\mathcal{R} = \mathcal{R}_0$
   - Each $\mathcal{R}_i$ is (N, K_i, D_i)
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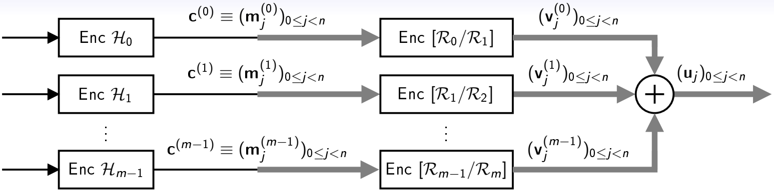

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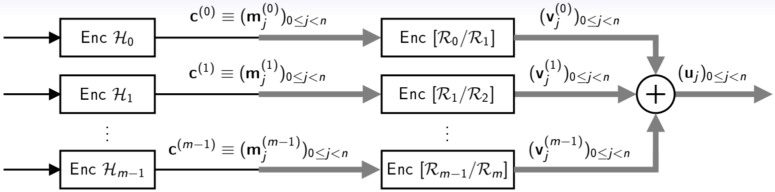

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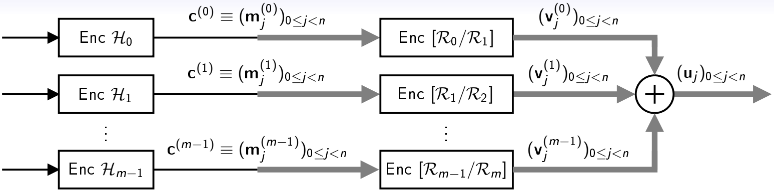

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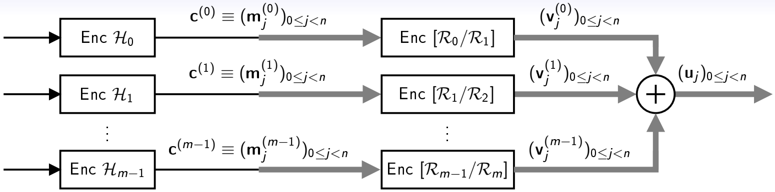

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Parent code partitioning

Recipe

1. Start with a Q-ary (N, K, D) parent code $\mathcal{R}$ with generator matrix

$$\mathbf{G} = \begin{bmatrix} g_{0,0} & g_{0,1} & \cdots & g_{0,K-1} \\ g_{1,0} & g_{1,1} & \cdots & g_{1,K-1} \\ \vdots & \vdots & \ddots & \vdots \\ g_{N-1,0} & g_{N-1,1} & \cdots & g_{N-1,K-1} \end{bmatrix}$$

2. Choose a decreasing sequence of integers: $K = K_0 > \cdots > K_m = 0$
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Example

$(N, K) = (6, 4) \qquad m = 3 \qquad (K_0, K_1, K_2, K_3) = (4, 3, 1, 0)$

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$[\mathcal{R}_0/\mathcal{R}_1]$ is generated by $\mathbf{G}[K_1 : K_0 - 1]$

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$[\mathcal{R}_1/\mathcal{R}_2]$ is generated by $\mathbf{G}[K_2 : K_1 - 1]$

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Code parameters

Result

A multishot rank-metric code $\mathcal{R} \subseteq (\mathbb{F}_{q^M}^N)^n$ with

$$|\mathcal{R}| = |\mathcal{H}_0| \cdots |\mathcal{H}_{m-1}|$$

and

$$d_{\min} \geq \min\{d_0 D_0, \ldots, d_{m-1} D_{m-1}\}$$

Proof.

Adaptation of Lin and Costello [Lin-Costello '04 (Chapter 15)]

□

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    - Originally for Euclidean metric codes (AWGN channel)
    - Starts with an m-level partitioning of a signal set (subset of $\mathbb{R}^2$)
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Effect of the parent code dimension

Assumptions:

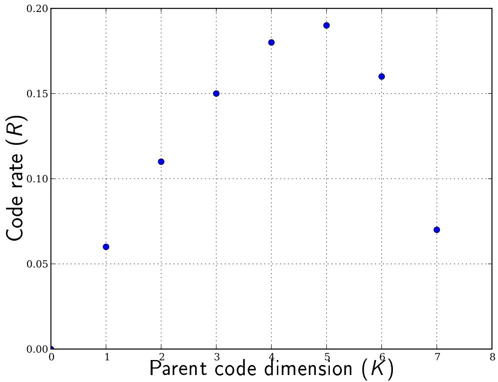
1. Partitioning is the "finest"
2. Rank metric codes $\mathcal{R}_i$ are MRD: Gabidulin codes
3. Component codes $\mathcal{H}_i$ are MDS: Reed-Solomon codes

How rate varies with parent code dimension K?

Effect of the parent code dimension

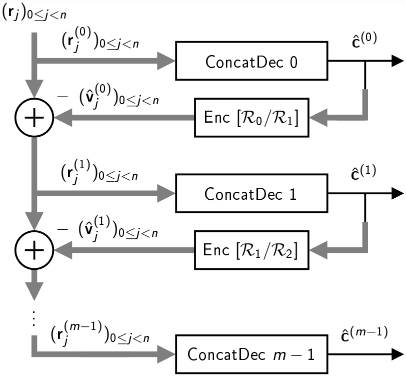
Binary: $q = 2$ Transmitted packets: $N = 8$ Packet size: $T = 40$, so $M = 32$

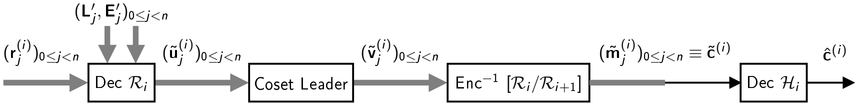
Number of shots: $n = 10$ Minimum distance: $d = \frac{1}{10} nT$



Decoding Procedure

Hard-decision multistage decoding





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This work:

- Multishot codes for non-coherent network coding
- Construction based on generalized concatenated coding
- Encoding and decoding procedures

Future work:

- How good are these codes?
- Soft-decision multistage decoding

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Thank you!

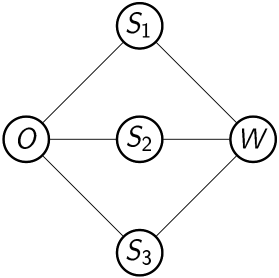
www.gpqcom.ufsc.br/~rwnobrega/

Backup Slides

Motivating example

Goal

A multishot code over $\mathcal{P}(\mathbb{F}_2^2)$ using the channel twice which is able to detect a single subspace error ($d = 2$)



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$$S_2 = \{00, 10\}$$
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Next: three proposals

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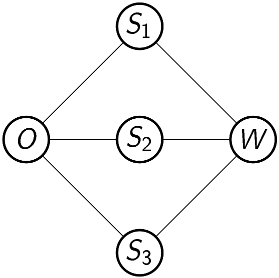

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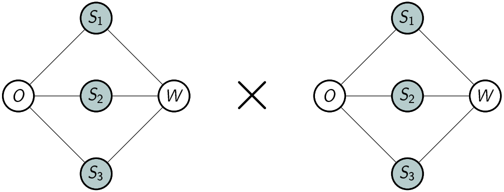
Next: three proposals

First attempt

Proposal 1

To extend the best 1-shot code



$$\mathcal{C}_1 = \{S_1S_1, S_1S_2, S_1S_3, S_2S_1, S_2S_2, S_2S_3, S_3S_1, S_3S_2, S_3S_3\}$$

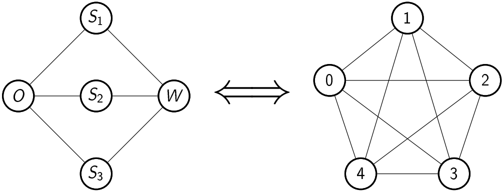
Cardinality

$$|\mathcal{C}_1| = 9$$

Second attempt

Proposal 2

Classic coding over the projective space



$$\mathcal{C}_2 = \{OO, S_1S_1, S_2S_2, S_3S_3, WW\}$$

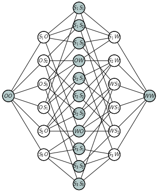
Cardinality

$$|\mathcal{C}_2| = 5$$

Third attempt

Proposal 3

To consider the metric space $\mathcal{P}(\mathbb{F}_2^2)^2$



$$\mathcal{C}_3 = \{OO, S_1S_1, S_1S_2, S_1S_3, OW, \ldots, S_3S_2, S_3S_3, WW\}$$

Cardinality

$$|\mathcal{C}_3| = 13$$

The metric space $\mathcal{P}(\mathbb{F}_2^2)^2$