On the Capacity of Multiplicative Finite-Field Matrix Channels

Roberto W. Nóbrega
Bartolomeu F. Uchôa-Filho
Danilo Silva

Federal University of Santa Catarina
Department of Electrical Engineering
Communications Research Group

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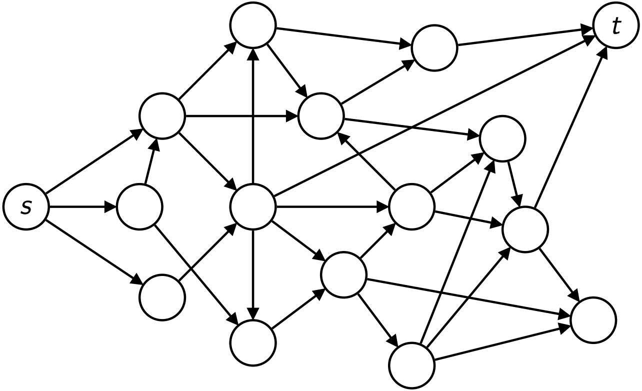
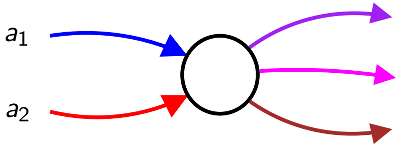
Motivation

Linear network coding



Linear network coding

- On each node: output is a linear combination of the input



$$a_3 = F_{31} a_1 + F_{32} a_2$$
$$a_4 = F_{41} a_1 + F_{42} a_2$$
$$a_5 = F_{51} a_1 + F_{52} a_2$$

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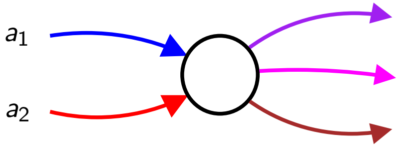

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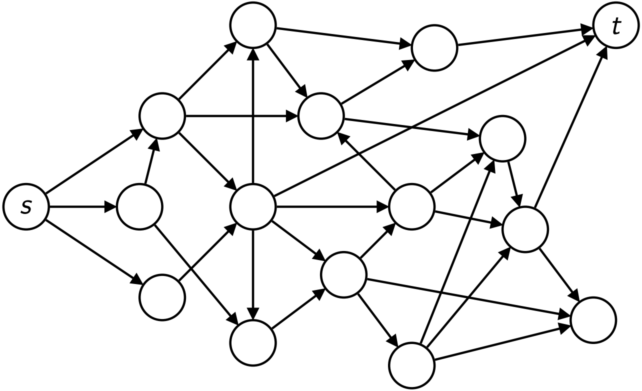
- Alphabet: finite field $\mathbb{F}_q$

End-to-end approach



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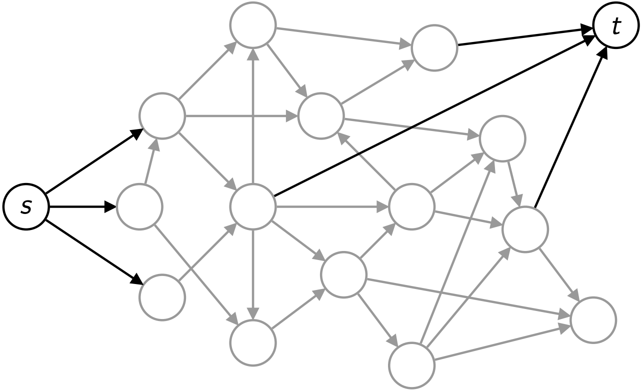

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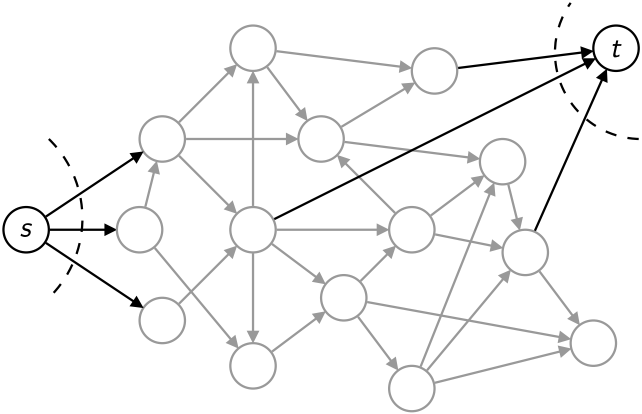

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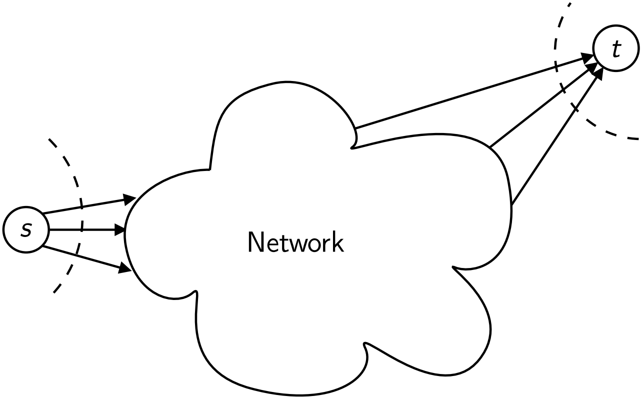

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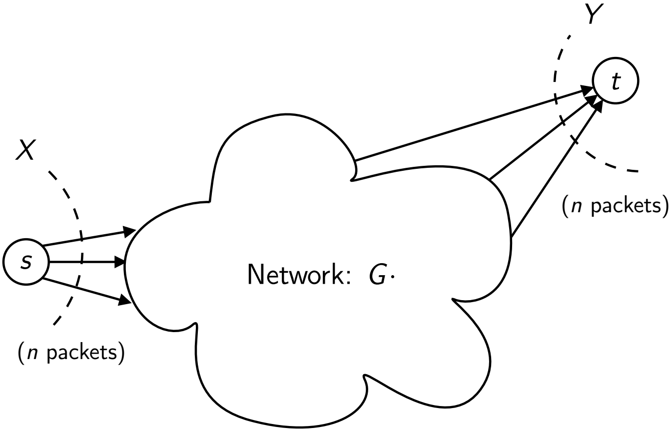
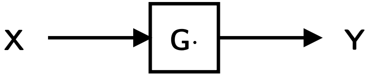


Packet length: m

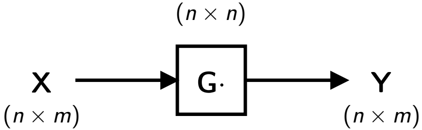
Channel model

Multiplicative finite-field matrix channels



$$Y = GX$$

Multiplicative finite-field matrix channels



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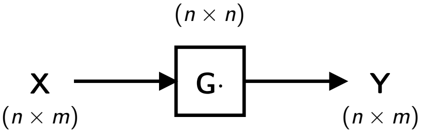

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- Probabilistic model: matrices are random variables (**bold**)

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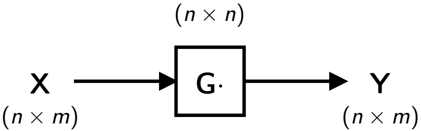

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- DMC defined by $(\mathcal{X}, p(Y|X), \mathcal{Y})$

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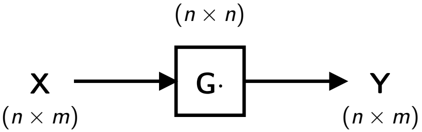
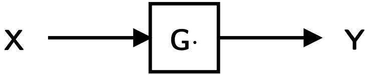


$$\mathbf{Y} = \mathbf{G}\mathbf{X}$$

- Probabilistic model: matrices are random variables
- DMC defined by $(\mathcal{X}, p(Y|X), \mathcal{Y})$
- $p(Y|X)$ induced by $p(G)$ through the channel law:

$$p(Y|X) = \sum_G p(G)\, 1[Y = GX]$$

Previous works



$$\mathbf{Y} = \mathbf{G}\mathbf{X}$$

- $\mathbf{G}$ full-rank, uniform [Silva et al., 2010]
- $\mathbf{G}$ with uniform i.i.d. entries $\equiv$ $\mathbf{G}$ uniform [Jafari et al., 2011]

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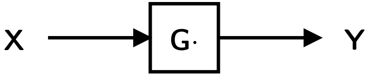

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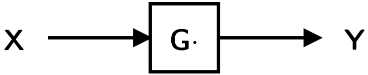

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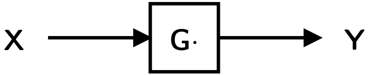

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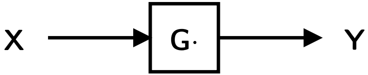
- **G** with arbitrary distribution [Yang et al., 2010]
  - Too general: complex channel description (q^{nm})

This work



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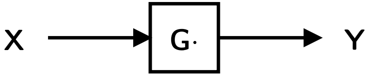

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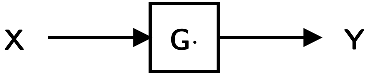

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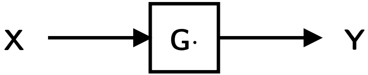
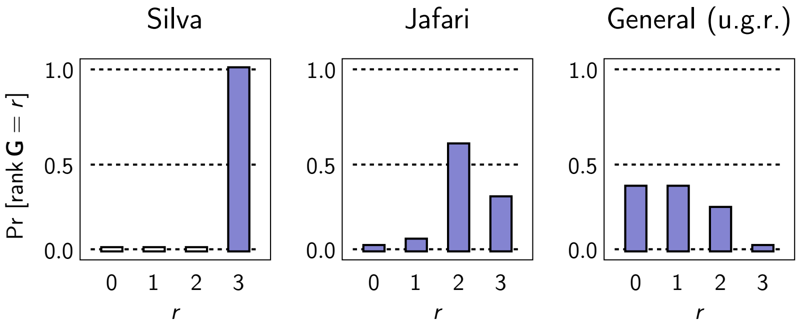


$$Y = GX$$

- **G** uniform given rank (u.g.r.) $\triangleq$ matrices with same rank are equiprobable
  - Simple channel description $(n + 1)$
  - Keeps the essence of non-coherence
  - Serves a lower bound on the capacity for the general case:

Comparison

- Example: $\mathbf{G}$ of dimension 3×3, binary field



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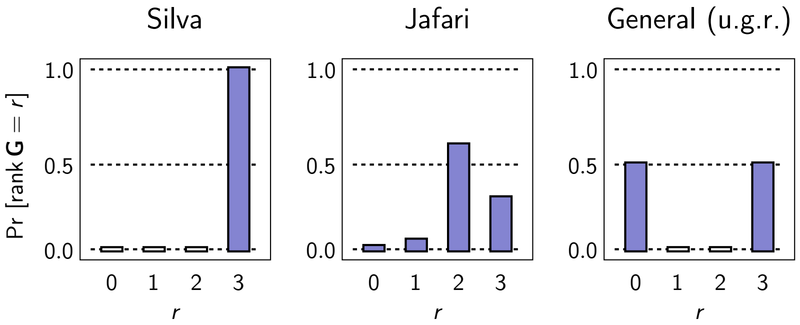

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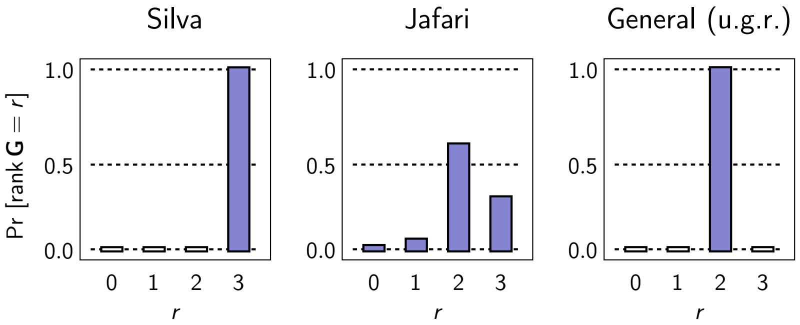

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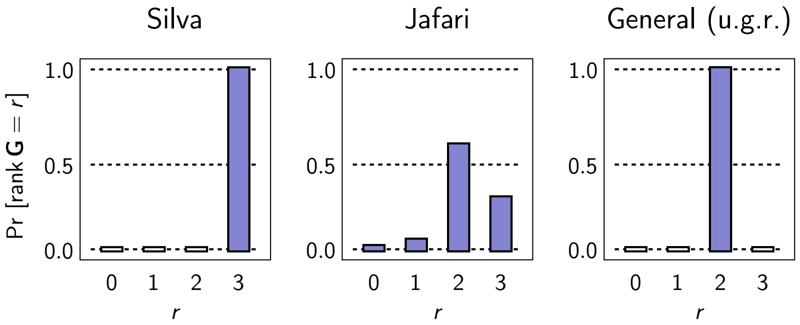

- Rank distribution depends on the topology, the link erasure probabilities, and the linear combinations

Our results

Channel transition probability

$$\mathbf{Y} = \mathbf{G}\mathbf{X}$$

$$\mathsf{u} \triangleq \operatorname{rank} \mathbf{X} \qquad \mathsf{v} \triangleq \operatorname{rank} \mathbf{Y} \qquad \mathsf{r} \triangleq \operatorname{rank} \mathbf{G}$$

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Channel transition probability

$$p(Y|X) = \begin{cases} \dfrac{p(v|u)}{|\mathcal{T}_q^{n\times u,v}|}, & \text{if } \langle Y \rangle \subseteq \langle X \rangle, \\ 0, & \text{else.} \end{cases}$$

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- $p(v|u)$: rank transition probability:

$$p(v|u) = \sum_{r=0}^{n} p(r) \frac{|\mathcal{T}_q^{n \times u, v}|}{|\mathcal{T}_q^{n \times n, r}|} \phi_q(n; u, n, v, r),$$

calculated using a combinatorial result [Brawley and Carlitz, 1973]

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- $p(Y|X)$: $q^{nm} \times q^{nm}$ matrix
- $p(v|u)$: $(n+1) \times (n+1)$ matrix

Rank transition probability

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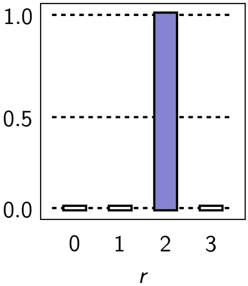
- Example: $\mathbf{G}$ of dimension 3×3, binary field

$p(r) = (0, 0, 1, 0)$



$$p(v|u) = \begin{bmatrix} u \backslash v & 0 & 1 & 2 & 3 \\ \hline 0 & 1 & 0 & 0 & 0 \\ 1 & \frac{1}{7} & \frac{6}{7} & 0 & 0 \\ 2 & 0 & \frac{3}{7} & \frac{4}{7} & 0 \\ 3 & 0 & 0 & 1 & 0 \end{bmatrix}$$

Channel capacity

- Maximization split in two stages:

$$C = \max_{p(X)} I(\mathbf{X}; \mathbf{Y})$$
$$= \max_{p(u)} \max_{p(X):p(u)} I(\mathbf{X}; \mathbf{Y})$$

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Inner stage: Solved, max achieved with u.g.r. input

$$\max_{p(X):p(u)} I(\mathbf{X};\mathbf{Y}) = \sum_{v} p(v) \log_q \frac{a_v}{p(v)} - \sum_{u} b_u p(u) \triangleq I^*(p(u))$$

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Outer stage: No closed-form solution

$$\max_{p(u)} I^*(p(u)) = \text{convex optimization problem}$$

Channel capacity

Original problem

$$C = \max_{p(X)} \; I(p(X))$$

Convex optimization on q^{nm} variables

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$$C = \max_{p(X)} \; I(p(X))$$

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New problem

$$C = \max_{p(u)} \; I^*(p(u))$$

Convex optimization on $n + 1$ variables

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Problems are equivalent

New problem

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Upper bound

#1 – Actual optimization problem

$$C = \max_{p(u)} \; I^*(p(u))$$

subject to $\quad p(0) + \ldots + p(n) = 1 \quad \forall u : p(u) \geq 0$

Upper bound

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#2 – Modified optimization problem

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Problems are **not** equivalent

#2 – Modified optimization problem

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Upper bound

- Exact solution for #2:

$$\tilde{C} = \log_q \sum_{v=0}^{n} |\mathcal{T}_q^{n\times m,v}| q^{-c_v}$$

where c_v's are the solution of a triangular linear system of equations

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- Upper bound: $C \leq \tilde{C}$
- Sometimes this is also the solution for #1!

Constant-rank input

- Input matrices limited to have rank u

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Rank-u capacity

$$C_u = \sum_{v=0}^{n} p(v|u) \log_q \frac{\begin{bmatrix} m \\ v \end{bmatrix}_q}{\begin{bmatrix} u \\ v \end{bmatrix}_q}$$

Constant-rank input

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Rank-u capacity

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- Asymptotically optimal: $\max_u C_u \leq C \leq \max_u C_u + \log_q n$

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Unconstrained capacity, for $q \to \infty$

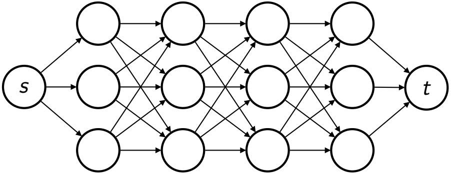
$$C = C_{u^*}$$

An example

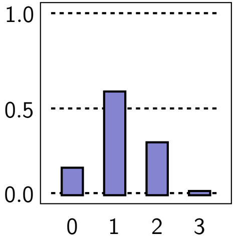
"Trellis network"

Pr[erasure] = 10% for each link, binary field

$n = 3 \quad m = 4$

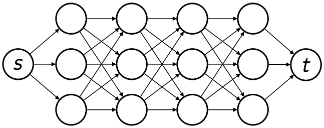


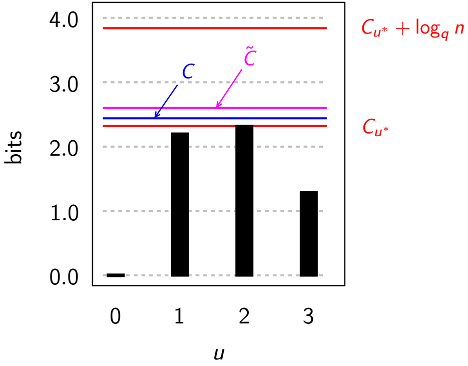
$p(r)$



An example

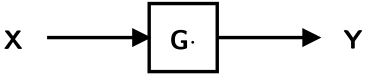
"Trellis network"





Communication via subspaces

- The matrix channel:



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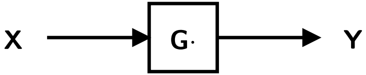
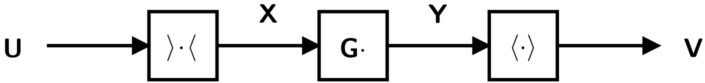


- The subspace channel:



Communication via subspaces

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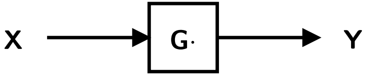

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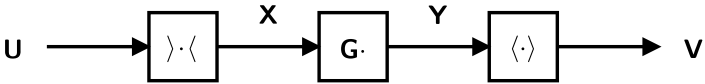

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$$I(\mathbf{X};\mathbf{Y}) = I(\mathbf{U};\mathbf{V})$$

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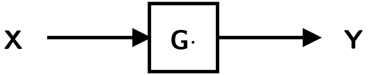

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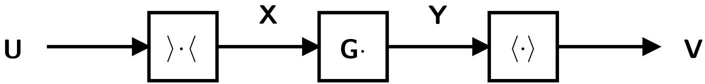

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- Approach: "grouping of letters" in a DMC

Conclusion

Review

- U.g.r. transfer matrix: reasonable model for non-coherent networks

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- Capacity: optimization problem, bounds, special cases

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Review

- U.g.r. transfer matrix: reasonable model for non-coherent networks
- Capacity: optimization problem, bounds, special cases
- Communication via subspaces: still optimal
- Main open problem: codes

Thank you!

Roberto W. Nóbrega
rwnobrega@eel.ufsc.br