On Multiplicative Matrix Channels over Finite Chain Rings

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  - Integers modulo n ($\mathbb{Z}_n$);
  - Quotients of Gaussian integers (e.g., $\mathbb{Z}_n[i]$);
  - Finite chain rings (including some of the above).
  - Products of those (e.g., $\mathbb{Z}_2 \times \mathbb{Z}_4$).

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    - Finite chain rings (including some of the above).
    - Products of those (e.g., $\mathbb{Z}_2 \times \mathbb{Z}_4$).
- Modules are the ring-theoretic counterpart of vector spaces.
    - Let R be a ring, and let Ω be a module over R.
    - Unlike vector spaces, we do not necessarily have $\Omega \cong R^n$.

Multiplicative Matrix Channels

Let R be a ring, and let n, m, ℓ be positive integers.

Definition

A multiplicative matrix channel (MMC) over R is a communication channel in which the input $\boldsymbol{X} \in R^{n \times \ell}$ and the output $\boldsymbol{Y} \in R^{m \times \ell}$ are matrices related by

$$\boldsymbol{Y} = \boldsymbol{A}\boldsymbol{X},$$

where $\boldsymbol{A} \in R^{m \times n}$ is called the transfer matrix.

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Physical-Layer Network Coding

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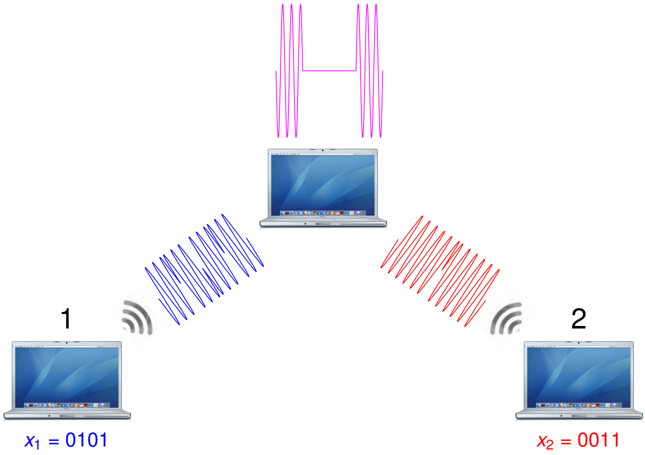
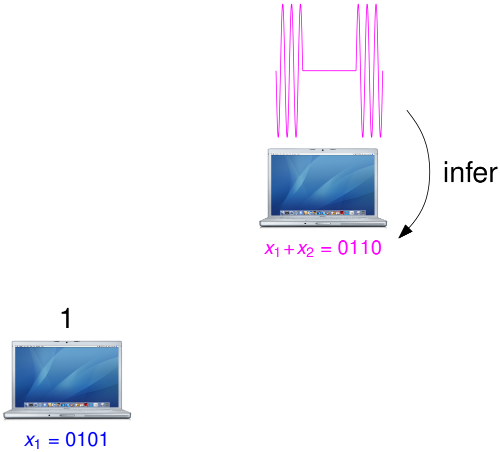


Physical-Layer Network Coding



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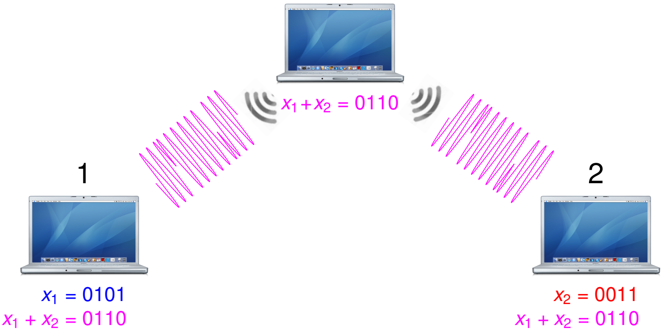


Physical-Layer Network Coding

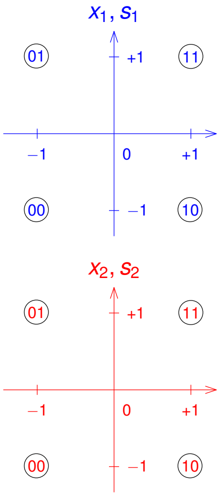
1

$x_1 = 0101$
$x_1 + x_2 = 0110$
$x_2 = 0011$

2

$x_2 = 0011$
$x_1 + x_2 = 0110$
$x_1 = 0101$

Example: QPSK Modulation [1]



$$\mathbb{Z}_2 \times \mathbb{Z}_2 = \{00, 10, 01, 11\}$$

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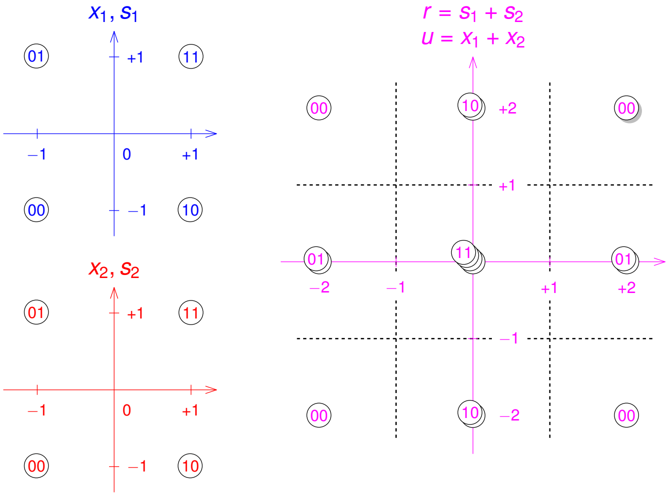

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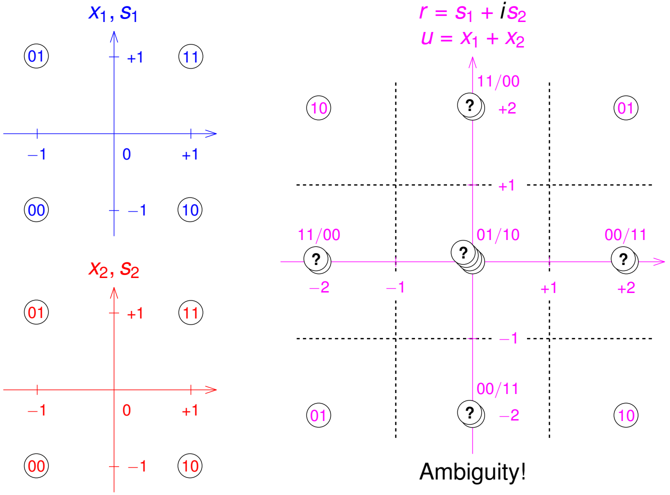
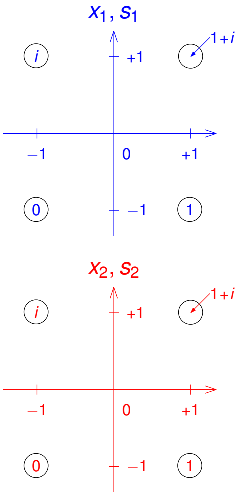


Example: QPSK Modulation [2]



Solution:

$$\mathbb{Z}_2[i] = \{0, 1, i, 1+i\}$$

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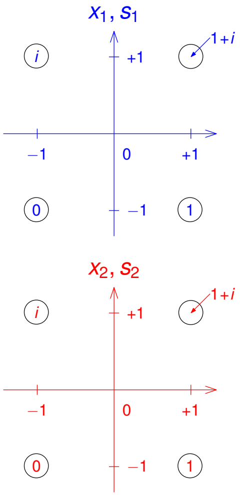

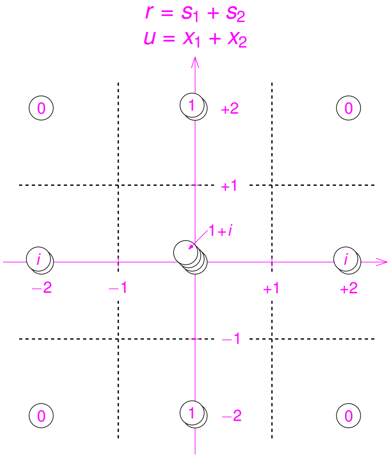

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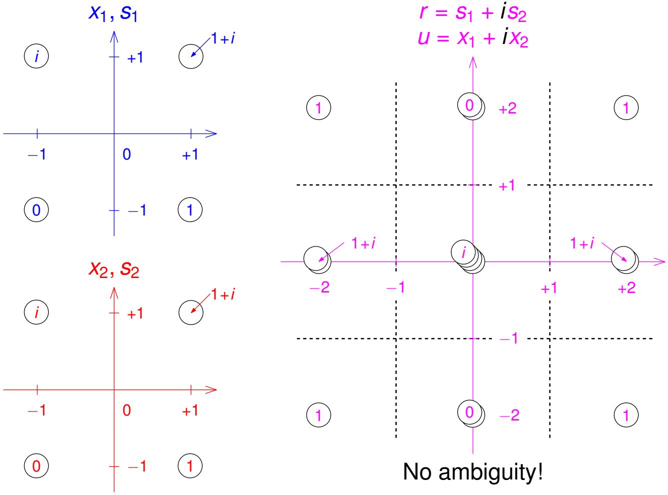

No ambiguity!

Rings and Modules Found in Practice

Uncoded modulation:

- 4-ASK $\longrightarrow$ $R = \mathbb{Z}_4$
- QPSK $\longrightarrow$ $R = \mathbb{Z}_2[i]$
- 16-QAM $\longrightarrow$ $R = \mathbb{Z}_4[i]$
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Beyond uncoded modulation: Nested-lattice-based PNC

- [Ordentlich, Zhan, Erez, Gastpar, Nazer, ISIT'11]
  Construction A applied to a binary LDPC code.
  $\longrightarrow$ $R = \mathbb{Z}_4$ and $\Omega = R^{54000} \times (2R)^{10800}$
- [Sakzad, Sadeghi, Panario, Allerton'10]
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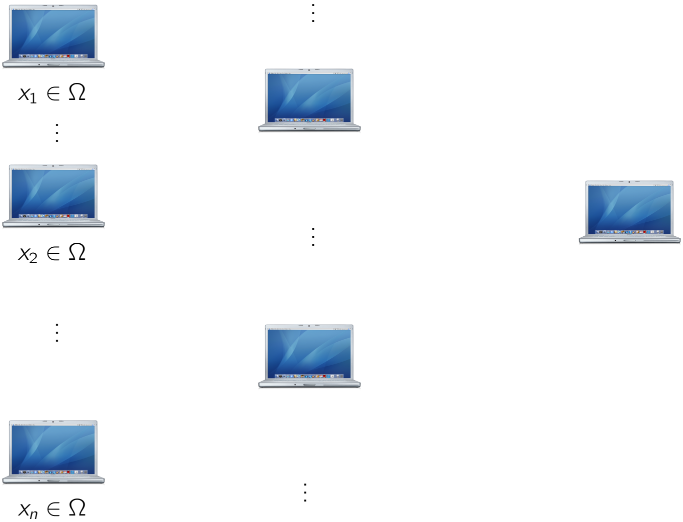
All of these are examples of finite chain rings.

Extension to Larger Networks



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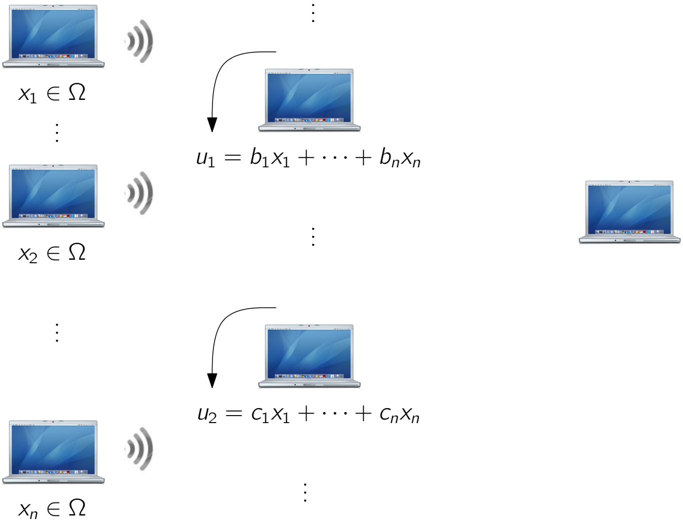

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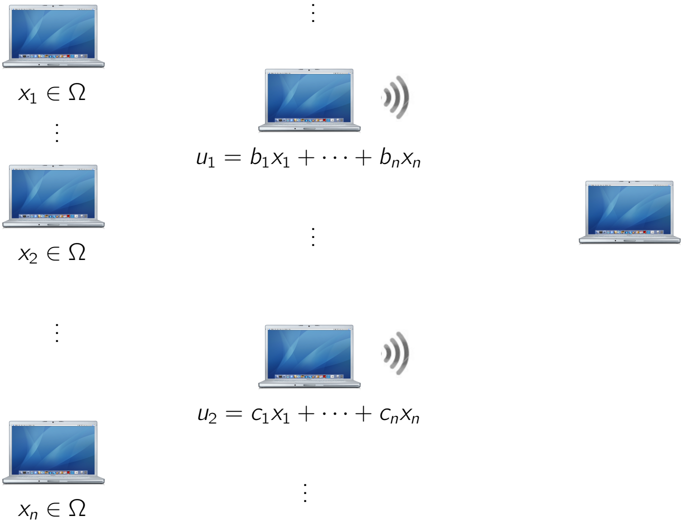

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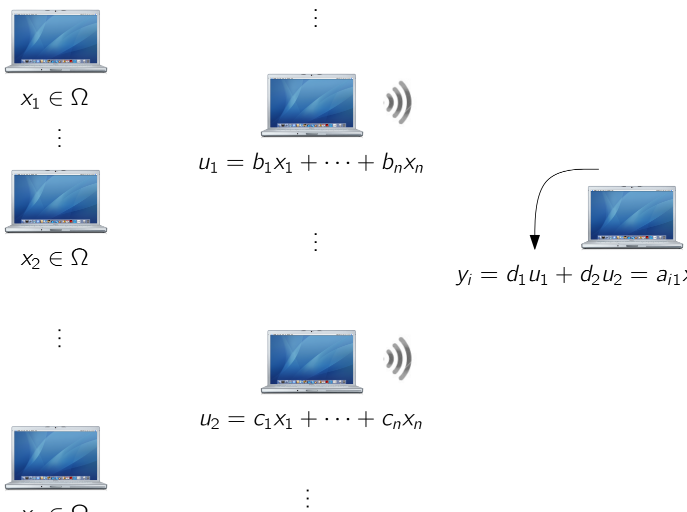

$x_1 \in \Omega$

$x_2 \in \Omega$

$x_n \in \Omega$

$u_1 = b_1 x_1 + \cdots + b_n x_n$

$u_2 = c_1 x_1 + \cdots + c_n x_n$

$y_i = d_1 u_1 + d_2 u_2 = a_{i1} x_1 + \cdots + a_{in} x_n$

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$$\begin{bmatrix} y_1 \\ \vdots \\ y_m \end{bmatrix} = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$$

$$Y = AX$$

Multiplicative matrix channel (MMC)

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Remark: Erroneous packets are discarded via linear error-detecting codes (over the ring).

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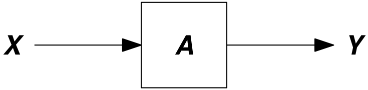
Multiplicative matrix channel (MMC)

The Problem

We study MMCs over finite chain rings.



Assumptions

- The probability distribution of $\boldsymbol{A}$ is arbitrary.
- $\boldsymbol{X}$ and $\boldsymbol{A}$ are independent.
- $\boldsymbol{A}$ is unknown at the transmitter, but known at the receiver (CSIR).

Contribution: Channel Capacity

Let s be the number of proper ideals of the finite chain ring.

Theorem

The channel capacity is achieved with uniform input and is given by

$$C = \sum_{i=0}^{s-1} \mathrm{E}[\rho_{s-i-1}]\lambda_i,$$

where $\lambda = \operatorname{shape} \Omega$, and $\rho = \operatorname{shape} \boldsymbol{A}$.

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The shape is an s-tuple of integers.

- The shape of a module generalizes the concept of dimension.
- The shape of a matrix generalizes the concept of rank.

Contribution: Coding Scheme

Overview of the coding scheme

We propose a coding scheme that adopts a layered approach by combining *s* codes over the residue field to obtain an overall code over the finite chain ring.

- The *code construction* makes use of the π-adic expansion.
- *Decoding* is performed in a multistage fashion, layer by layer.

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Code features

- Capacity-achieving;
- Polynomial time complexity;
- Universal: only the knowledge of $\mathrm{E}[\rho]$ is needed ($\rho =$ shape $\boldsymbol{A}$).

Thank you!

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